

A NOTE ON WATER-CLOCKS AND ON THE
AUTHORITY OF TEXTS

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In his *Altägyptische Zeitmessung*, Ludwig Borchardt [1920: 15f] explains that the outflow velocity v of water from a water-clock is $v = \mu\sqrt{2gh}$, where h is the water height over the outlet, g the acceleration of gravity, and μ a reduction factor which takes account of friction and of the narrowing of the jet that occurs when it leaves the opening (without the factor μ , the formula is simply Torricelli's theorem). As a consequence (he does not give the simple derivation, but it is correct provided μ does not depend on h) he states that the water level descends uniformly if the clock is a paraboloid of the fourth order, i.e., if the radius of the water surface is proportional to $\sqrt[4]{h}$.

In his discussion of the ratio 2:1 of the longest to the shortest day given in MUL.APIN (II.i.10,24, ed. [Hunger & Pingree 1989: 72, 75]), Neugebauer [1947: 39] explains this impossible claim from the fact that "the time t needed to empty [a cylindrical vessel of height h through a whole in the bottom] completely is [...] given by $t = c\sqrt{h}$, where c is constant depending on the outlet and the area of cross section of the cylinder". Even Neugebauer does not explain how the formula is derived, but it follows from the same velocity formula. The argument is repeated in shorter form in [Neugebauer 1975: 708].

Subsequent authors refer to Borchardt's and Neugebauer's calculations – see, e.g., [Hunger & Pingree 1989: 151], [Bremner 1993: 373], and [Clagett 1995: 76]. It seems that nobody has ever checked the calculation or the underlying model. Since a simple experiment shows the model to be misleading (due to surface tension the clock will never run empty, as anybody provided with a plastic bottle for mineral water and a needle may find out), it may be worthwhile tracing its origin and limitations.

Borchardt does not tell from where he has taken his formula, but his choice of symbols shows that he has consulted the standard literature on hydrodynamics – [Prandtl 1913: 126] may well have been his source. Here it becomes clear that the formula – used in technical hydraulics – is meant for water leaving through an opening in a wall not considerably thicker than the diameter of the opening (this is why the jet contracts), and that the way friction is taken into account as a constant factor on v is empirical and crude. The whole approximation is meant for cases where surface tension is negligible and friction almost so.

In order to see whether these assumptions are justified we may look at the idealized water-clock in Figure 1 (so far it does not matter whether it is cylindrical, conical, or of any other shape, as long as the actual water surface A is much larger than the cross-section of the outlet, which we assume to be cylindrical with radius r . ℓ designates the length of the outlet cylinder, which for simplicity we may assume to be 1 cm – the true value will probably have been somewhat greater (in the Egyptian specimens, 2–4 cm, see [Borchardt 1920: Tafel 2, 5, 6]), and the effect of friction thus even larger than what we are going to find. ρ is the density of water ($\rho \approx 1 \text{ g cm}^{-3}$), η its viscosity ($\eta \approx 1 \cdot 10^{-2} \text{ g cm}^{-1} \text{ sec}^{-1}$)

at 20° C), α its surface tension under air ($\alpha \approx 72 \text{ g sec}^{-2}$ at 20° C).¹

In Babylonian water-clocks, 1 mina $\approx 500 \text{ g}$ of water is supposed to flow out in 4 hours; the amount of water leaving per second is thus $\sigma_b \approx 3.5 \cdot 10^{-2} \text{ g sec}^{-1}$; in the Karnak clock, it is somewhat greater, $\sigma_k \approx 0,8 \text{ g sec}^{-1}$. In order to know whether the flow is turbulent or laminar, we should see whether Reynold's number $Re = \frac{vr\rho}{\eta}$ exceeds the critical value 1000 or not (see, e.g., [Prandtl 1913: 119]); since

$$\sigma = \pi r^2 v \rho \quad (*)$$

we find $Re = \frac{\sigma}{\pi \eta r} \approx 32 \frac{\sigma}{r}$. In the Babylonian clock we thus have $Re_b \approx \frac{1.1}{r}$, while the Karnak clock gives $Re_k \approx \frac{26}{r}$. We may assume the flow to be laminar in a Babylonian clock as long as $r > 1.1 \cdot 10^{-3} \text{ cm}$,

while the condition for the Karnak clock is $r > 0.26 \text{ mm}$. As we shall see, the former condition is amply fulfilled, while the latter is somewhat problematic.

For the moment we shall assume the outflow to be laminar; then the drop of pressure Δp in the outlet is determined (approximately, considering ℓ "very long") from Poiseuille's law,

$$\sigma = \rho \pi r^2 v = \rho \frac{\Delta p}{\ell} \frac{\pi r^4}{8 \eta}$$

whence

$$\Delta p = \frac{8 \ell \eta v}{r^2}$$

corresponding to a reduction of the water height

$$h_r = \frac{\Delta p}{\rho g} = \frac{8 \ell \eta v}{\rho g r^2}$$

In the time interval dt , an amount of water

$$dm = \sigma dt = \rho \pi r^2 v dt = -A \rho dh$$

streams out of the clock. This corresponds to a loss of potential energy (corrected for friction)

$$E_p = (h - h_r) g dm = (h - h_r) g \rho \pi r^2 v dt$$

Part of this energy is converted into the kinetic energy of the corresponding section of the jet,

$$E_k = \frac{1}{2} v^2 dm = \frac{1}{2} \rho \pi r^2 v^3 dt,$$

and the remainder is taken up by the surface energy of the same cylindrical section,

$$E_s = 2 \pi r v \alpha dt$$

Since $E_p = E_k + E_s$, we therefore have

$$\left(h - \frac{8 \ell \eta}{\rho g r^2} v\right) \cdot g r^2 \rho = \frac{1}{2} \rho r^2 v^2 + 2 r \alpha \quad (**)$$

or, rearranging,

$$g r^2 \rho \cdot \left(h - \frac{2 \alpha}{g r \rho}\right) = \frac{1}{2} \rho r^2 v^2 + 8 \ell \eta v$$

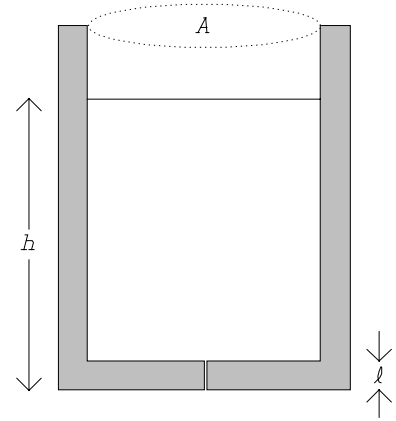


Figure 1

¹ All physical constants are taken from [Kaye & Laby 1959]. Since the dimensions of the ancient water-clocks are (at best, i.e., when the clocks themselves have survived) only determined with restricted precision, I have rounded all values in the interest of readability of the formulae. Units are in the c-g-s-system.

We see immediately that $v = 0$ when $h = h_s = \frac{2\alpha}{gr\rho}$, i.e., at the moment when the surface tension $2\frac{\alpha}{r}$ of a hemisphere spanned over the outlet balances the remaining water height h_s .

If H designates the water height over this minimum level, i.e., $H = h - h_s$, the formula becomes

$$gH = \frac{1}{2}v^2 + \frac{8\ell\eta}{r^2\rho}v \quad (\dagger)$$

If friction can be neglected, i.e., if

$$\frac{16\ell\eta}{r^2\rho} \ll v \ll \frac{gHr^2\rho}{8\ell\eta} \quad (\ddagger)$$

this reduces to

$$gH = \frac{1}{2}v^2$$

i.e., Torricelli's theorem. Neugebauer's formula for the emptying time will hence be correct, provided, firstly, that we do not ask for emptying but for termination of the outflow, and secondly, *that the model be valid and friction can be disregarded almost until this very moment*. Even in the quasi-absence of friction, however, Borchartd's claim that the water level descends uniformly if the clock is a paraboloid of the fourth order is only valid if the outflow takes place at a level well below the bottom of this paraboloid, viz, if $\ell = h_s$. If friction cannot be disregarded (*and it cannot for water-clocks of the dimensions we speak about*, the double condition (\ddagger) will not be fulfilled), Borchartd's linear approximation $v = \mu\sqrt{2gH}$ to the effects of friction only holds for small relative variations of H , for which reason the fourth-order paraboloid is anyway unable to provide a uniform descent of the water level. Nor does Neugebauer's formula for the emptying time hold good: If we disregard friction and calculate v from $gH = \frac{1}{2}v^2$, putting $H = 25$ cm and $\ell = 1$ cm, we find $v \approx 220$ cm sec⁻¹. With $\sigma_b \approx 3.5 \cdot 10^{-2}$ g sec⁻¹ this leads to $r_b \approx 7 \cdot 10^{-3}$ cm, and to

$$\frac{8\ell\eta v}{r^2\rho} \approx 14gH$$

instead of the presupposed

$$\frac{8\ell\eta v}{r^2\rho} \ll gH$$

The water level h_s carried by the surface tension will be c. 21 cm.²

This shows that friction is much too important to be neglected; in order to estimate the real values of r and v we will have to combine (*) and (\dagger) , which gives

$$gH = \left(\frac{1}{2} + \frac{8\pi\ell\eta}{\sigma}\right) \cdot v^2$$

Inserting $\sigma_b \approx 3.5 \cdot 10^{-2}$ g sec⁻¹, we find $v \approx 57$ cm sec⁻¹, $r \approx 0.14$ mm, whence $h_s \approx 10.3$ cm. Even with the provisos for the validity of the model to which we shall return, Babylonian water-clocks would thus retain a conspicuous amount of water. For the Karnak clock, with $\sigma_k \approx 0.8$ g sec⁻¹, we find $v \approx 176$ cm sec⁻¹, $r \approx 0.38$ mm, and $h_s \approx 3.8$ cm.

It might seem attractive to use the differential equation (\dagger) in order to get an exact description of the process, in the way Torricelli's theorem with or without correction factor has been applied so far. There are two reasons that this cannot be done. Firstly, the outflow of the water does not stop at

² For the Karnak outflow $\sigma_k \approx 0.8$ g sec⁻¹, we get $r_k \approx 3.4 \cdot 10^{-2}$; the neglected member is c. $0.6gh$ (better, yet still too large), and $h_s \approx 4.4$ cm; Re_k will be c. 750, disturbingly close to the critical limit.

the height $H = 0$, $h = h_s$. The reason is adhesion. Instead of standing in a hemisphere with diameter $2r$ over the opening, the water creeps over the surface, allowing the formation of a larger drop, which eventually falls. This disturbance of the jet starts already well before the water level h equals h_s – the Karnak clock, indeed, drips much of the time [Clagett 1995: 69]. The exact influence of adhesion, however, depends on the geometry of the opening and on the material of the surface around it; the only way to know precisely the rate at which ancient water-clocks were emptied is to measure experimentally on surviving specimens, hoping that devices used to calibrate the flow have not been lost since the time when the clocks were used.³

Even this, however, is a method with limited validity. At 10° C (no impossible night temperature in Mesopotamia), the viscosity of water is $1.3 \text{ g cm}^{-1} \text{ sec}^{-1}$; at 40° C (no impossible day temperature) it is $0.65 \text{ g cm}^{-1} \text{ sec}^{-1}$, exactly the half; since friction dominates the right-hand side of the equation (†) for clocks of the Babylonian type, this introduces a major uncertainty.

We are thus left without means to explain from the physics of a water-clock why the compiler of MUL.APIN would give a value for the ratio between the longest and the shortest day which even with the techniques of the day could be seen to be far removed from truth. But we probably need no physical explanation. If some of the finest intellects of the present century have repeated Torricelli's theorem or the statements of Borchardt and Neugebauer without ever checking their adequacy either theoretically or experimentally, there is little reason to wonder or be scandalized if a Babylonian scholar-astronomer did the same thing some 3000 years ago. He may just have been a very modern mind.

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³ Cf. [Clagett 1995: 149 n. 86], which quotes Horapollon's *Hieroglyphica* for the information that a small iron plug was used for this purpose in Egypt. This probably concerns inflow clocks, but it is not unlikely that outflow clocks were calibrated in a similar way, or by manipulating the orifice of the metal tube which was used at least in Egypt.

Since the Reynold's number in the Karnak clock was already close to the critical value, any control device or manipulation of the orifice might risk to provoke turbulence, making thus the whole idealized model even less realistic.